Sturm-Liouville Theory

1-D Heat eq: $u_t = k u_{xx}$ $\quad 0 < x < L$

we've seen that if $u(0,t) = u(L,t) = 0$ (ends frozen)

then $\lambda_n = \frac{n^2\pi^2}{L^2}$ with $X_n = \sin(\sqrt{\lambda}\, x) = \sin\left(\frac{n\pi}{L} x\right)$ $\quad n = 1, 2, 3, \ldots$

if $u_x(0,t) = u_x(L,t) = 0$ (ends insulated)

then $\lambda_n = \frac{n^2\pi^2}{L^2}$ with $X_n = \cos(\sqrt{\lambda}\, x) = \cos\left(\frac{n\pi}{L} x\right)$ $\quad n = 0, 1, 2, 3, \ldots$

in both cases, the eigenvalues λ_n give us the frequencies of the modes of solutions → all integer multiples of $\frac{\pi}{L}$

and the eigenfunctions are all mutually orthogonal

$$\int_0^L X_n X_m \, dx = 0 \quad \text{if } m \neq n$$

is this always the case w/ other kinds of boundary conditions?

let's look at this one : $u_t = k u_{xx}$ $\quad 0 < x < L$

$u(0,t) = 0$ $\quad$ left end frozen

$\underbrace{u_x(L,t) = -h\, u(L,t)}$ $\quad h > 0$

models a heat exchange at the right end

(from Newton's Law of Cooling)

after separating the variables, we get to

$$X'' + \lambda X = 0$$

$$X(0) = 0$$

$$X'(L) + h X(L) = 0$$

after using $X(0) = 0$, we get $X = B \sin(\sqrt{\lambda}\, x) \longrightarrow X_n = \sin(\sqrt{\lambda_n}\, x)$

$$X' = \sqrt{\lambda}\, B \cos(\sqrt{\lambda}\, x)$$

$$X'(L) + h X(L) = 0 \rightarrow \sqrt{\lambda}\, B \cos(\sqrt{\lambda}\, L) + h B \sin(\sqrt{\lambda}\, L) = 0$$

need: $\lambda \neq 0$, $B \neq 0$

$$\sqrt{\lambda}\cos(\sqrt{\lambda}L) = -h\sin(\sqrt{\lambda}L)$$

$$\boxed{\tan(\sqrt{\lambda}L) = -\frac{\sqrt{\lambda}}{h}}$$ solve for λ

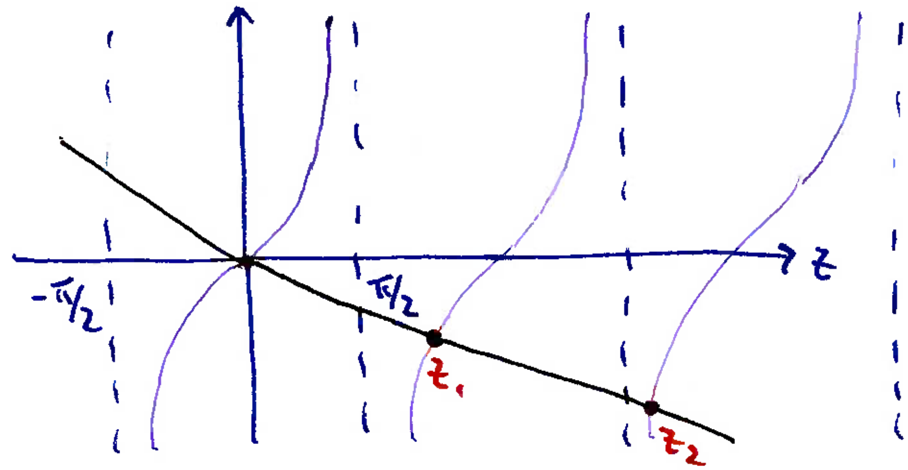
this is a transcendal equation (variable we are interested in is on both sides and cannot be isolated)

the graphical interpretation: let $z = \sqrt{\lambda}L$

that equation becomes $\tan(z) = -\frac{z}{hL}$ $(h>0,\ L>0)$

→ z is the intersection of $\tan(z)$ and the line

$$-\frac{z}{hL}$$



we want to find all positive intersections $z = z_1, z_2, \ldots$

whatever z_n is, it is clearly NOT integer multiples of $\frac{\pi}{L}$

$X_n = \sin(\sqrt{\lambda_n}\, x)$ frequencies are not $\frac{n\pi}{L}$ anymore

as $n \to \infty$, $z_n \to$ the left asympote of each tangent cycle

$\rightarrow$ eventually resembling one of the two basic cases what if diffusivity is not constant? $u_t = k(x)\,u_{xx}$?

Sturm-Liouville theory gives us a big picture understanding of the solutions to $X'' + \lambda X = 0$

Sturm-Liouville Problem

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + q(x)\,y + \lambda\, w(x)\,y = 0 \qquad a < x < b$$

subject to BCs

$\alpha_1 y(a) + \alpha_2 y'(a) = 0$ $\quad$ α_1, α_2 not both zero

$\beta_1 y(b) + \beta_2 y'(b) = 0$ $\quad$ β_1, β_2 not both zero

β_1

notice if $p=1,\ q=0,\ w=1$ we get $y'' + \lambda y = 0$ (Fourier)

$p = x,\ q = -\frac{n^2}{x},\ w = x$ we get $xy'' + y' + \left(\lambda x - \frac{n^2}{x}\right)y = 0$

Solutions are Bessel functions (waves of a circular drum)

$p = 1-x^2,\ q = 0,\ w = 1$ we get $y''(1-x^2) - 2xy' + \lambda y = 0$

Solutrons are Legendre polynomials
(steady state solution of a heated sphere)

big picture: different p, q, w → different heat/wave eq. situations